Inst.: Data- och informationsteknik
Kursnamn: Logic in Computer Science
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All answers and solutions must be carefully motivated!
total 60; ≥28: 3, ≥38: 4, ≥50: 5
total 60; ≥28: G, ≥42: VG

All answers **must** be carefully motivated.

1. We consider the following language: we have one binary predicate symbol R and one constant c.

   (a) Define what a model of this language is. (3p)
   Solution: A model $\mathcal{M}$ for this language is:

      i. A nonempty set A,
      ii. a set $R^{\mathcal{M}} \subseteq A^2$,
      iii. and a constant element $c^{\mathcal{M}} \in A$.

   (b) Give an example of a formula in this language which does not hold in all models. (2p)
   Solution: The formula $\varphi = R(c, c)$ does not hold in the model $\mathcal{M}$ given by:

      i. $A = \{0, 1\}$,
      ii. $R^{\mathcal{M}} = \{(0, 0)\}$,
      iii. $c^{\mathcal{M}} = 1$.

   With this model and an arbitrary look-up table l we get $\mathcal{M} \not\models_l \varphi$ as $(1, 1) \notin R^{\mathcal{M}}$.

2. Give proofs in natural deduction of the following sequents:

   (a) $\vdash (\neg p \to p) \to \neg\neg p$ (3p)
   Solution:

1	$\neg p \to p$	Assumption
2	$\neg p$	Assumption
3	p	$\to_e$ 1, 2
4	$\bot$	$\neg_e$ 3, 2
5	$\neg\neg p$	$\neg_i$ 2 − 4
6	$(\neg p \to p) \to \neg\neg p$	$\to_i$ 1 − 5

(b) $\vdash (p \to q) \lor (q \to r)$ (3p)

Solution:

1	$q \lor \neg q$	LEM
2	q	Assumption
3	p	Assumption
4	q	copy 2
5	$p \to q$	$\to_i$ $3-4$
6	$(p \to q) \lor (q \to r)$	$\lor_{i_1}$ 5
7	$\neg q$	Assumption
8	q	Assumption
9	$\bot$	$\neg_e$ $8, 7$
10	r	$\bot_e$ 9
11	$q \to r$	$\to_i$ $8-10$
12	$(p \to q) \lor (q \to r)$	$\lor_{i_2}$ 11
13	$(p \to q) \lor (q \to r)$	$\lor_e$ $1, 2-6, 7-12$

(c) $\neg p \to q,\ r \to p \vdash \neg q \lor r \to p$ (3p)

Solution:

1	$\neg p \to q$	Premise
2	$r \to p$	Premise
3	$\neg q \lor r$	Assumption
4	$\neg q$	Assumption
5	$\neg\neg p$	MT $1, 4$
6	p	$\neg\neg_e$ 5
7	r	Assumption
8	p	$\to_e$ $2, 7$
9	p	$\lor_e$ $3, 4-6, 7-8$
10	$\neg q \lor r \to p$	$\to_i$ $3-9$

3. Compute a conjunctive normal form (CNF) of the formula:

$$\neg(r \to p \vee q) \vee (\neg p \to q \wedge \neg r) \quad \text{(3p)}$$

Solution: A CNF of the formula is:

$$(p \vee \neg q \vee \neg r) \wedge (p \vee q \vee r)$$

4. Give proofs in natural deduction of the following sequents:

(a) $\forall x \, \forall y \, (P(y) \to Q(x)) \vdash \exists y \, P(y) \to \forall x \, Q(x)$ (3p)
Solution:

1	$\forall x \, \forall y \, (P(y) \to Q(x))$	Premise
2	$\exists y \, P(y)$	Assumption
3	x_0	
4	$\forall y \, (P(y) \to Q(x_0))$	$\forall x_e$ 1
5	$y_0 \quad P(y_0)$	Assumption
6	$P(y_0) \to Q(x_0)$	$\forall_e$ 4
7	$Q(x_0)$	$\to_e$ 6, 5
8	$Q(x_0)$	$\exists y_e$ 2, 5 − 7
9	$\forall x \, Q(x)$	$\forall x_i$ 3 − 8
10	$\exists y \, P(y) \to \forall x \, Q(x)$	$\to_i$ 2 − 9

(b) $c_1 = c_2 \vee d_1 = d_2 \vdash f(c_1) = f(c_2) \vee f(d_1) = f(d_2)$ (3p)
Solution:

1	$c_1 = c_2 \vee d_1 = d_2$	Premise
2	$c_1 = c_2$	Assumption
3	$f(c_1) = f(c_1)$	$=_i$
4	$f(c_1) = f(c_2)$	$=_e$ 2, 3
5	$f(c_1) = f(c_2) \vee f(d_1) = f(d_2)$	$\vee_{i_1}$ 4
6	$d_1 = d_2$	Assumption
7	$f(d_1) = f(d_1)$	$=_i$
8	$f(d_1) = f(d_2)$	$=_e$ 6, 7
9	$f(c_1) = f(c_2) \vee f(d_1) = f(d_2)$	$\vee_{i_2}$ 8
10	$f(c_1) = f(c_2) \vee f(d_1) = f(d_2)$	$\vee_e$ 1, 2 − 5, 6 − 9

(c) $\forall x\, P(x) \wedge \forall y\, Q(y) \vdash \forall x\, (P(x) \wedge Q(x))$ (3p)
Solution:

1	$\forall x\, P(x) \wedge \forall y\, Q(y)$	Premise
2	$\forall x\, P(x)$	$\wedge_{e_1}$ 1
3	$\forall y\, P(y)$	$\wedge_{e_2}$ 1
4	x_0	
5	$P(x_0)$	$\forall x_e$ 2
6	$Q(x_0)$	$\forall y_e$ 3
7	$P(x_0) \wedge Q(x_0)$	$\wedge_i$ 5, 6
8	$\forall x\, (P(x) \wedge Q(x))$	$\forall x_i$ 4 − 7

5. Is the following LTL formula valid, i.e., satisfied on all paths of all transition systems?

$$\mathrm{G}(p \vee q) \rightarrow \mathrm{G}\,\mathrm{F}\, p \vee \mathrm{G}\,\mathrm{F}\, q \quad \text{(5p)}$$

Solution: Yes it is valid. Let π be a path in a model $\mathcal{M}$ such that $\pi \models \mathrm{G}(p \vee q)$. We now have to show that $\pi \models \mathrm{G}\,\mathrm{F}\, p \vee \mathrm{G}\,\mathrm{F}\, q$. Assume $\pi \not\models \mathrm{G}\,\mathrm{F}\, p$ and $\pi \not\models \mathrm{G}\,\mathrm{F}\, q$ then there are i and j such that $\pi^i \models G\neg p$ and $\pi^j \models G\neg q$, so for all $k \geqslant max(i, j)$ we have $\pi^k \models \neg p \wedge \neg q$. But this contradicts that $\pi \models \mathrm{G}(p \vee q)$. Hence $\pi \models \mathrm{G}\,\mathrm{F}\, p \vee \mathrm{G}\,\mathrm{F}\, q$.

6. Let P and Q be unary predicates. Decide for each of the sequents below whether they are valid or not, i.e., give a proof in natural deduction or a counter-model.

(a) $\exists x \neg P(x) \vee \exists x\, Q(x) \vdash \forall x\, (P(x) \rightarrow Q(x))$ (3p)
Solution: This is not valid, a counter-model $\mathcal{M}$ is given by:

- $A = \{0, 1\}$,
- $P^{\mathcal{M}} = \{0\}$,
- $Q^{\mathcal{M}} = \{1\}$.

With an arbitrary look-up table l we have $\mathcal{M} \models_l \exists x \neg P(x) \vee \exists x Q(x)$ as $1 \notin P^{\mathcal{M}}$ (or as $0 \in Q^{\mathcal{M}}$), but $\mathcal{M} \not\models_l \forall x\, (P(x) \rightarrow Q(x))$ as $0 \in P^{\mathcal{M}}$ but $0 \notin Q^{\mathcal{M}}$. So by *soundness* we get $\exists x \neg P(x) \vee \exists x\, Q(x) \nvdash \forall x\, (P(x) \rightarrow Q(x))$.

(b) $\exists x\,(P(x) \to Q(x)) \vdash \exists x\, P(x) \to \exists x\, Q(x)$ (3p)

Solution: This is not valid, a counter-model $\mathcal{M}$ is given by:

- $A = \{0, 1\}$,
- $P^{\mathcal{M}} = \{0\}$,
- $Q^{\mathcal{M}} = \emptyset$,

With an arbitrary look-up table l we have $\mathcal{M} \models_l \exists x\,(P(x) \to Q(x))$ as $1 \notin P^{\mathcal{M}}$ which makes the implication true. But $\mathcal{M} \not\models_l \exists x\, P(x) \to \exists x\, Q(x)$ as even though $0 \in P^{\mathcal{M}}$, $\exists x\, Q(x)$ is always false as $Q^{\mathcal{M}}$ is empty. So by *soundness* we get $\exists x\,(P(x) \to Q(x)) \nvdash \exists x\, P(x) \to \exists x\, Q(x)$.

(c) $\forall x\,(P(x) \to Q(x)) \vdash \exists x P(x) \to \exists x Q(x)$ (3p)

Solution: This is valid, proof:

1	$\forall x\,(P(x) \to Q(x))$	Premise
2	$\exists x\, P(x)$	Assumption
3	$x_0 P(x_0)$	Assumption
4	$P(x_0) \to Q(x_0)$	$\forall x_e$ 1
5	$Q(x_0)$	$\to_e$ 4, 5
6	$\exists x\, Q(x)$	$\exists x_i$ 6
7	$\exists x\, Q(x)$	$\exists x_e$ 2, 3 − 6
8	$\exists x P(x) \to \exists x Q(x)$	$\to_i$ 2 − 7

7. Consider the transition system $\mathcal{M} = (S, \to, L)$ where the states are $S = \{s_0, s_1, s_2, s_3\}$, the transitions are $s_0 \to s_1, s_0 \to s_3, s_1 \to s_1, s_2 \to s_0, s_2 \to s_1, s_2 \to s_2$, and $s_3 \to s_2$, and the labeling function is given by $L(s_0) = \{r\}$, $L(s_1) = \{p, q\}$, $L(s_2) = \{r\}$, $L(s_3) = \{p, r\}$.

   (a) Which are the states s that satisfy the CTL formula $\mathrm{AF}\, p$ (i.e., where $\mathcal{M}, s \models \mathrm{AF}\, p$)? (2p)

   Solution: s_0, s_1 and s_3. (These must be motivated by analysing the each state separately using the definition of satisfiability for CTL from the book)

   (b) Do we have $\mathcal{M}, s_2 \models \mathrm{A}[r \,\mathrm{U}\, q]$? (2p)

   Solution: No. The path $\pi = s_2 \to s_2 \to s_2 \to \ldots$ is a counter-example.

(c) Do we have $\mathcal{M}, s_0 \models \mathrm{EG}(p \to \mathrm{AX}\,\neg p)$? (2p)
Solution: Yes, take the path $\pi = s_0 \to s_3 \to s_2 \to s_2 \to \ldots$

(d) Explain why the LTL formula $\mathrm{F}\,\mathrm{G}\,\neg r \to \mathrm{F}\,\mathrm{G}\,p$ is satisfied on every path in $\mathcal{M}$. (2p)
Solution: The only state where we have $\neg r$ is s_1 and any path satisfying $\mathrm{F}\,\mathrm{G}\,\neg r$ must go there at some point i. But then it will be stuck there and $\mathrm{G}\,p$ would hold at i and hence does $\mathrm{F}\,\mathrm{G}\,p$ holds on any path satisfying $\mathrm{F}\,\mathrm{G}\,\neg r$.

8. We consider the language with one binary predicate symbol R and a unary function symbol f. Consider the formulas:

$$\begin{aligned} \varphi_1 &= \forall x\, R(x, f(x)), \\ \varphi_2 &= \forall x\, \forall y\, \forall z\, (R(x,y) \wedge R(y,z) \to R(x,z)), \\ \varphi_3 &= \forall x\, \neg R(x,x). \end{aligned}$$

(a) Give a model in which $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is true. (3p)
Solution: Define $\mathcal{M}$ by:

- $A = \mathbb{N}$,
- $R^{\mathcal{M}} = \{(x,y) \mid x < y\}$,
- $f^{\mathcal{M}}(x) = x + 1$,
- l arbitrary as all formulas are closed.

This gives that:

i. $\mathcal{M} \models_l \varphi_1$ as for all $a \in \mathbb{N}$ we have $a < a + 1$.
ii. $\mathcal{M} \models_l \varphi_2$ as $<$ is transitive.
iii. $\mathcal{M} \models_l \varphi_3$ as $<$ is irreflexive.

(b) Show that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ cannot have a finite model. (5p)
Solution: Let $\mathcal{M}$ be a model of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$. Since the domain is nonempty there is an $a \in A$. Since $\mathcal{M} \models \varphi_1$ we get $(a, g(a)) \in R^{\mathcal{M}}$, $(g(a), g^2(a)) \in R^{\mathcal{M}}$, $\ldots$, $(g^n(a), g^{n+1}(a)) \in R^{\mathcal{M}}$, $\ldots$ where $g \stackrel{def}{=} f^{\mathcal{M}}$. Assume that there exists $n < m$, $n, m \in \mathbb{N}$ such that $g^n(a) = g^m(a)$. Since $R^{\mathcal{M}}$ is transitive (since $\mathcal{M} \models \varphi_2$) and $(g^n(a), g^{n+1}(a)) \in R^{\mathcal{M}}$, $(g^{n+1}(a), g^{n+2}(a)) \in R^{\mathcal{M}}$ we get $(g^n(a), g^{n+2}(a)) \in R^{\mathcal{M}}$. Continuing this way we get $(g^n(a), g^m(a)) \in R^{\mathcal{M}}$. But this contradicts $\mathcal{M} \models \varphi_3$. Hence all the $g^n(a)$, $n \in \mathbb{N}$ are distinct and $\mathcal{M}$ must have infinitely many elements.

9. A set of connectives is called *adequate* if for every formula of propositional logic there is an equivalent formula using only connectives from this set.

Explain why $\{\wedge, \vee\}$ is *not* an adequate set of connectives. (4p)
Solution: It is not possible to express $\neg$ using only $\wedge$, $\vee$ and propositional atoms as any combination of these will be true in a valuation assigning true to all of the involved atoms. But for $\neg$ the value is false when the involved atom is true.

Alternative solution: It is not possible to express $\rightarrow$ using only $\wedge$, $\vee$ and propositional atoms as any combination of these will be false in a valuation assigning false to all of the involved atoms. But for $\rightarrow$ the value is true when the involved atoms are all false.

Good Luck!

Anders, Jan, and Simon